

On Elliptic Differential Operators with Shifts

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1 Statement of the Problem and Main Theorem

The motivating point of our research was differential operators on the noncommutative torus¹ studied by Connes [1, 2], who in particular obtained an index formula for such operators. These operators include shifts (more precisely, in this case, irrational rotations); hence our interest in general differential equations with shifts naturally arose.

Let M be a smooth closed manifold. We consider operators on M of the form

$$Du = \sum_j (g_j^*)^{-1} (D_j u), \quad (1)$$

where the sum is finite, D_j are differential operators, and g_j^* are shift operators,

$$g_j^* u(x) = u(g_j(x)). \quad (2)$$

Here $g_j: M \rightarrow M$ are some diffeomorphisms. Elliptic theory for operators of the form (1) is well known (e.g., see [3]). In particular, under certain assumptions about g_j , the principal symbol of the operator (1) is defined and its invertibility implies that the operator (1) is Fredholm in appropriate Sobolev spaces on M . However, apart from Connes' example, the index formula for such operators has so far been obtained only for the case in which the group Γ generated by the elements g_j is *finite*, and neither the formula itself nor the proof method make sense for Γ infinite. However, it is the case of an *infinite* group that is of main interest in applications, as, for example, in Connes' above-mentioned work. In the present paper, we prove an index formula for the case of an *infinite* group Γ satisfying certain mild assumptions.

¹Which have important applications to specific physical problems such as quantum Hall effect.

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1.1 Definition of the operator algebra

1.1.1 The group Γ

We assume that M is a compact oriented Riemannian manifold and Γ is a finitely generated dense subgroup of a compact Lie group of orientation-preserving isometries of M satisfying the following two conditions:

1. (The *polynomial growth* condition.) For some finite system of generators of Γ , the number of distinct elements of Γ representable by words of length $\leq k$ in these generators grows not faster than some power of k .

In what follows, we fix some system of generators and denote by $|g|$ the minimum length of words representing $g \in \Gamma$.

2. (The “*Diophantine*” condition.) There exists a finite N and a constant $C > 0$ such that the estimate

$$\text{dist}(g(x), x) \geq C|g|^{-N} \text{dist}(x, \text{fix}(g))$$

holds for all $x \in M$ and $g \in \Gamma$. Here $\text{dist}(\cdot, \cdot)$ stands for the Riemannian distance between points of the manifold, and $\text{fix}(g)$ is the set of fixed points of $g \in \Gamma$. If $\text{fix}(g)$ is empty, then by convention we set $\text{dist}(x, \text{fix}(g)) = 1$.

Each element $g \in \Gamma$ naturally acts on the cotangent space T^*M (it takes each fiber T_x^*M to the fiber $T_{g(x)}^*$ by the linear transformation $((dg)^*)^{-1}$, where dg is the differential of g); this action will be denoted by ∂g . Since g is an isometry, it follows that ∂g is also an isometry and in particular has a well-defined restriction to S^*M .

1.1.2 Pseudodifferential operators

Let D_g , $g \in \Gamma$, be a family of pseudodifferential operator of order $\leq m$ on M rapidly (i.e., faster than any power of $|g|^{-1}$) decaying as $|g| \rightarrow \infty$ in the Fréchet topology on the set of pseudodifferential operators (e.g., see [4]). In particular, it follows that

$$\|D_g: H^s(M) \rightarrow H^{s-m}(M)\| \leq C_{sN}(1 + |g|)^{-N}$$

for any s and N . Then the series

$$D = \sum_{g \in \Gamma} (g^*)^{-1} \circ D_g : H^s(M) \rightarrow H^{s-m}(M) \quad (3)$$

converges absolutely in operator norm for every $s \in \mathbb{R}$. The set of such operators will be denoted by Ψ_Γ^m , and we set

$$\Psi_\Gamma^\infty = \bigcup_m \Psi_\Gamma^m.$$

One can readily prove that under our assumptions Ψ_Γ^∞ is an algebra. It is called the *algebra of Ψ DO with shifts* on M .

1.1.3 Symbols and the Fredholm property

For the operator (3), we define its (*principal*) *symbol* by the formula

$$\sigma(D) := \sum_{g \in \Gamma} (\partial g^*)^{-1} \circ \sigma(D_g) : L^2(S^*M) \longrightarrow L^2(S^*M), \quad (4)$$

where $\sigma(D_g)$ is the conventional principal symbol of the m th-order pseudodifferential operator D_g , acting as a multiplication operator.

Theorem 1 ([3]). *The principal symbol is well defined, and the operator*

$$D : H^s(M) \longrightarrow H^{s-m}(M),$$

where $D \in \Psi_\Gamma^m$, is Fredholm if and only if its symbol $\sigma(D)$ is invertible. Its index is independent of s .

The expression (4) belongs to the algebra $C^\infty(S^*M)_\Gamma$ of $C^\infty(S^*M)$ -valued functions on Γ rapidly decaying in the Fréchet topology of $C^\infty(S^*M)$ as $|g| \rightarrow \infty$. By [5, 6], this is a dense local subalgebra in $C(S^*M)_\Gamma \equiv C(S^*M) \rtimes \Gamma$. Consequently, the inverse of an elliptic symbol $\sigma \in C^\infty(S^*M)_\Gamma$ also belongs to $C^\infty(S^*M)_\Gamma$.

1.2 Main Theorem

1.2.1 Notation

To state the index theorem, we need some notation.

If E is a vector bundle over a manifold, then by $\lambda_{-1}(E)$ we denote the (virtual) vector bundle

$$\lambda_{-1}(E) := \sum_{j \geq 0} (-1)^j \Lambda^j(E) = \Lambda^{\text{even}}(E) - \Lambda^{\text{odd}}(E),$$

composed of the exterior powers of E (the sum is actually finite). By

$$\mathrm{ch} E(g) := \mathrm{tr} \left(g^* \exp \left(-\frac{1}{2\pi i} \Omega \right) \right)$$

we denote the localized Chern character of a bundle E with some fixed curvature form Ω at an element $g \in \Gamma$. Here tr stands for the trace in the fibers of the vector bundle.

Next, let us define the Chern–Simons character of an elliptic element $a \in C^\infty(S^*M)_\Gamma$. Consider the algebra $\Lambda(S^*M)_\Gamma$ of $\Lambda(S^*M)$ -valued functions on Γ rapidly decaying in the Fréchet topology of $\Lambda(S^*M)$ (with the obvious multiplication). This algebra is graded (by form degree) and becomes a differential graded algebra if we equip it with the differential

$$d \left(\sum_g (\partial g^*)^{-1} \circ \omega_g \right) := \sum_g (\partial g^*)^{-1} \circ (d\omega_g). \quad (5)$$

Now we can define the Chern–Simons character by setting

$$\begin{aligned} \mathrm{ch} a &:= \mathrm{ch}_1 a + \mathrm{ch}_3 a + \dots, \quad \text{where} \\ \mathrm{ch}_{2k+1} a &:= \left(\frac{1}{2\pi i} \right)^{k+1} \frac{(k)!}{(2k+1)!} \mathrm{tr}[a^{-1} da]^{2k+1}. \end{aligned} \quad (6)$$

By $\mathrm{ch} a(g)$ we denote the differential form that is the coefficient of g^* in the expansion

$$\mathrm{ch} a = \sum_{g \in \Gamma} (\partial g^*)^{-1} \circ \mathrm{ch} a(g).$$

Next, recall that if g is an isometry of M , then the set $\mathrm{fix} g$ of fixed points of g is a disjoint union of smooth submanifolds of M [7], each of which is locally (i.e., in a neighborhood of its every point) the image, under the geodesic exponential mapping, of the eigenspace of dg corresponding to the eigenvalue 1. These submanifolds will be denoted by M_g . (We shall not use double subscripts to avoid clumsiness.) Our index formula involves integration over the cosphere bundles S^*M_g ; we equip these with the standard orientation coming from the almost complex structure on T^*M_g . By NM_g we denote the normal bundle of M_g .

1.2.2 The index formula

Theorem 2. *Let $D \in \Psi^\infty(M, \mathrm{Mat}(m, \mathbb{C}))_\Gamma$ be a matrix elliptic operator on M . Then the following index formula holds:*

$$\mathrm{ind} D = \sum_{g \in \Gamma} \left[\sum_{M_g} \int_{S^*M_g} \frac{\mathrm{Td}(T^*M_g \otimes \mathbb{C})}{\mathrm{ch} \lambda_{-1}(NM_g \otimes \mathbb{C})(g)} \mathrm{ch} \sigma(D)(g) \right]. \quad (7)$$

Here the denominators do not vanish, and the double series converges absolutely.

Remark 3. The nonvanishing of the denominators was proved in [7]. However, the convergence of the series is still to be proved.

2 Proof of the Index Formula

Let $D \in \Psi^\infty(M, \text{Mat}(m, \mathbb{C}))_\Gamma$ be a matrix elliptic operator on M . For simplicity, we assume that D is a first-order differential operator. The general case can be treated by order reduction and by using the technique of operators with continuous symbols, just as in the classical paper [8].

Our proof is based on the reduction of the original operator on M to a Dirac type operator on S^*M with the use of a trick that is apparently due to Kasparov. (However, we have not been able to find anything of this sort in the literature and so cannot give a precise reference.)

2.1 Reduction to an operator on the cosphere bundle

We first perform reduction to the cotangent bundle and only then use some K -theory to go down to the cosphere bundle.

2.1.1 Elliptic theory on the cotangent bundle

We consider symbols $\sigma(x, \xi, p, \eta)$, where (x, ξ) are standard coordinates on T^*M and (p, η) are the dual momenta, satisfying the estimates

$$|\partial_x^\alpha \partial_z^\beta \sigma| \leq C_{\alpha\beta} (1 + |z|)^{m-|\beta|},$$

where $z = (\xi, p, \eta)$. We also introduce associated principal symbols, homogeneous in z . The corresponding operators are considered in spaces with the norms

$$\|u\|_s^2 := \int_{T^*M} |(\Delta_x + \Delta_\xi + \xi^2)^{s/2} u|^2 dx d\xi,$$

where Δ_x and Δ_ξ are positive Laplace operators in the corresponding variables. The ellipticity condition is that the principal symbol should be invertible for $z \neq 0$.

The group Γ acts on T^*M by the isometries ∂g . Hence one can define, in a manner similar to the above, an algebra of Ψ DO with shifts on T^*M , which will be denoted by $\Psi(T^*M)_\Gamma$.

2.1.2 Reduction to the cotangent bundle

In the space \mathbb{R}^n , $n = \dim M$, consider the elliptic operator

$$E = [(x + \partial/\partial x)dx] \wedge + [(x - \partial/\partial x)dx] \lrcorner : C^\infty(\mathbb{R}^n, \Lambda^{ev}(\mathbb{C}^n)) \longrightarrow C^\infty(\mathbb{R}^n, \Lambda^{odd}(\mathbb{C}^n)),$$

where

$$x dx := \sum_j x_j dx_j; \quad \frac{\partial}{\partial x} dx := \sum_j \frac{\partial}{\partial x_j} dx_j.$$

This operator is elliptic of index one in appropriate Sobolev spaces; its kernel is one-dimensional and is spanned by the function $\exp(-x^2/2)$. Moreover, the operator is $O(n)$ -invariant. Hence we can consider the family $\mathcal{E} = \{\mathcal{E}_x\}_{x \in M}$ of such operators acting in the fibers of the cotangent bundle,

$$\mathcal{E} : C^\infty(T^*M, \pi^* \Lambda^{ev}(M)) \longrightarrow C^\infty(T^*M, \pi^* \Lambda^{odd}(M)), \quad \pi : T^*M \longrightarrow M.$$

In turn, the operator D can be lifted to the operator

$$D \otimes 1_\Lambda : C^\infty(T^*M, \mathbb{C}^m \otimes \pi^* \Lambda(M)) \longrightarrow C^\infty(T^*M, \mathbb{C}^m \otimes \pi^* \Lambda(M)).$$

Lemma 4. *The crossed product*

$$D \# \mathcal{E} := \begin{pmatrix} D \otimes 1_{\Lambda^{ev}} & -1_{\mathbb{C}^m} \otimes \mathcal{E}^* \\ 1_{\mathbb{C}^m} \otimes \mathcal{E} & D^* \otimes 1_{\Lambda^{odd}} \end{pmatrix} : C^\infty(T^*M, \mathbb{C}^m \otimes \pi^* \Lambda(M)) \longrightarrow C^\infty(T^*M, \mathbb{C}^m \otimes \pi^* \Lambda(M))$$

is an elliptic operator in $\Psi(T^*M)_\Gamma$, and one has

$$\text{ind } D = \text{ind}(D \# \mathcal{E}).$$

Proof. The proof is by analogy with that of a similar assertion in [8]. \square

2.1.3 Reduction to the Todd operator

We continue the symbol of D as a first-order homogeneous function to the entire space T^*M .

Lemma 5. *One has*

$$\text{ind}(D \# \mathcal{E}) = \text{ind}(\sigma(D) \# \mathcal{D}), \tag{8}$$

where \mathcal{D} is the Todd operator on the space T^*M , viewed as an almost complex manifold with complex coordinates

$$z_1 = \xi_1 + ix_1, \dots, z_n = \xi_n + ix_n.$$

Proof. We denote the variables on T^*M by (x, ξ) and the dual momenta by (p, η) .

The symbols of the operators in (8) belong to the algebra

$$C^\infty(S(T^*M \oplus T^*M \oplus T^*M))_\Gamma,$$

where Γ acts on the cosphere bundle of T^*M by the second differential

$$\partial^2 g := \partial(\partial g) : T^*(T^*M) \longrightarrow T^*(T^*M).$$

Let $D = \sum D_g g^*$. Then the symbols of the operators on the left- and right-hand sides in (8) are

$$\begin{aligned} & \left(\sum_g (\partial^2 g^*)^{-1} \circ \sigma(D_g)(x, p) \right) \# \sigma(\mathcal{E})(\xi, \eta), \\ & \left(\sum_g (\partial^2 g^*)^{-1} \circ \sigma(D_g)(x, \xi) \right) \# \sigma(\mathcal{E})(-p, \eta), \end{aligned}$$

respectively. These symbols can be joined by the homotopy

$$\left(\sum_g (\partial^2 g^*)^{-1} \circ \sigma(D_g)(x, \xi \sin \varphi + p \cos \varphi) \right) \# \sigma(\mathcal{E})(\xi \cos \varphi - p \sin \varphi, \eta),$$

$\varphi \in [0, \pi/2]$, and one can readily verify that this homotopy preserves ellipticity. The proof of the lemma is complete. \square

2.1.4 Passage to the cosphere bundle

The Todd operator on T^*M induces the so-called *tangential Todd operator*

$$\mathcal{D}_{S^*M} : C^\infty(S^*M, \Lambda^{ev}(M)) \longrightarrow C^\infty(S^*M, \Lambda^{ev}(M))$$

on the cosphere bundle $S^*M \subset T^*M$. This is an elliptic symmetric operator. To describe its principal symbol, we identify the tangent and cotangent bundle using the metric and take local coordinates (x, ξ, p, η) on $T^*(T^*M)$. Let $c(p, \eta) := \sigma(\mathcal{D})(x, \xi, p, \eta)$, where \mathcal{D} is the Todd operator; then the symbol of the tangential Todd operator is given by

$$\sigma(\mathcal{D}_{S^*M})(x, \xi, p, \eta) := ic(0, \xi)c(p, \eta),$$

where ξ , $|\xi| = 1$, is a unit covector and $(p, \eta) \in T^*_{(x, \xi)} S^*M$, $\eta \perp \xi$, is a covector tangent to S^*M .

A standard K -theoretic argument shows that the following lemma is true.

Lemma 6. *One has*

$$\text{ind} [\sigma(D) \# \mathcal{D}] = \text{ind} [P\sigma(D)|_{S^*M}P],$$

where

$$P\sigma(D)|_{S^*M}P : PL^2(S^*M, \Lambda^{ev}(M) \otimes \mathbb{C}^m) \longrightarrow PL^2(S^*M, \Lambda^{ev}(M) \otimes \mathbb{C}^m) \quad (9)$$

is a Toeplitz operator on the subspace determined by the positive spectral projection P of the tangential Todd operator on the cosphere bundle.

Thus, to compute the index of the original operator D , it remains to compute the index of the Toeplitz operator (9) on S^*M .

2.2 Computation of the index of a Dirac type operator

The Todd operator that arose in the preceding subsection is a Dirac type operator with noncommuting coefficients. In this section, we show how to compute the index of such operators. First, we deal with the case of even-dimensional manifolds, and then use the standard technique of multiplication by \mathbb{S}^1 to cover the case of S^*M , which is actually of interest to us.

2.2.1 The even case

Let X be a smooth closed manifold on which the group Γ acts isometrically. We assume that the power growth and Diophantine conditions are satisfied and that the manifold is even-dimensional and is equipped with a spin structure, which is moreover preserved by the action of Γ . By

$$D_+ : C^\infty(X, S_+) \longrightarrow C^\infty(X, S_-)$$

we denote the corresponding Γ -invariant Dirac operator on X . This operator is local with respect to the action of the algebra $C^\infty(X)_\Gamma$, and hence we obtain an elliptic operator if we twist D_+ by some projection

$$p \in \text{Mat}(N, C^\infty(X)_\Gamma)$$

over the algebra $C^\infty(X)_\Gamma$ of rapidly decaying $C^\infty(X)$ -valued functions on the group.

We denote the twisted operator by

$$pD_+p : pC^\infty(X, S_+ \otimes \mathbb{C}^N) \longrightarrow pC^\infty(X, S_- \otimes \mathbb{C}^N). \quad (10)$$

To give the index formula, we introduce the noncommutative Chern character of the projection p . Consider the graded differential algebra $\Lambda(X)_\Gamma$. The noncommutative Chern character is given by

$$\text{ch } p := \text{tr } p \left[\exp \left(-\frac{1}{2\pi i} p d p d p \right) \right] \in \Lambda(X)_\Gamma,$$

where, as above, tr is the usual matrix trace.

Theorem 7. *One has*

$$\operatorname{ind} pD_+p = \sum_{g \in \Gamma} \left[\sum_{X_g \subset X} \int_{X_g} A(X_g) \operatorname{Pf} \left\{ 2i \sin \left(\frac{\Omega}{4\pi} + \frac{i\Theta}{2} \right) (NX_g) \right\}^{-1} \cdot \operatorname{ch} p(g) \right], \quad (11)$$

where the double sum is absolutely convergent. Here

- X_g are connected components of the set of fixed points of g , and each X_g is a smooth even-dimensional manifold;
- $\Omega(NX_g)$ is the curvature form of the normal bundle NX_g ;
- $\Theta(NX_g)$ is the logarithm of Jacobi's matrix of the mapping $g^* : NX_g \rightarrow NX_g$;
- Pf is the Pfaffian of a skew-symmetric matrix;
- $\operatorname{ch} p(g)$ is the coefficient of $(g^*)^{-1}$ in the decomposition of the Chern character as an element of $\Lambda(X)_\Gamma$;
- $A(X_g)$ is a differential form representing the A -class of the manifold X_g .

Proof. For simplicity, we assume that the Dirac operator D_+ is invertible. The proof is based on the application of the Connes–Moscovici formula for the computation of the Chern–Connes character [9] of the operator D_+ with the subsequent computation of the terms in this formula with the use of the explicit formulas given in [10, Theorem 5, p. 471]. These computations are standard but clumsy, and we omit them. Instead, we show that the Connes–Moscovici formula applies to our case.

Let $D : C^\infty(X, S_+ \oplus S_-) \longrightarrow C^\infty(X, S_+ \oplus S_-)$ be the full Dirac operator

$$D = \begin{pmatrix} 0 & (D_+)^* \\ D_+ & 0 \end{pmatrix}.$$

By the results given in [11], the Connes–Moscovici formula applies if the following two conditions are satisfied for the spectral triple $(C^\infty(X)_\Gamma, L^2(X, S), D)$ determined by the Dirac operator:

1) It has *finite analytic dimension*; i.e., for some $d, q \in \mathbb{R}$ and all differential operators $P \in \Psi_\Gamma^\infty$ the function $\operatorname{tr} P\Delta^{-z}$ is holomorphic in the half-plane $\operatorname{Re} z > (\operatorname{ord} P - q)/d$ (here $\Delta := D^*D$).

2) It has *the analytic continuation property*; i.e., for each differential operator $P \in \operatorname{Diff}(X)_\Gamma^\infty$ the function $\operatorname{tr} P\Delta^{-z}$ admits a meromorphic continuation into the entire complex plane.

The first assumption obviously holds. (It is valid for operators without shifts, and for operators with shifts the proof is the same.)

Let us verify the second assumption. Let $P = \sum_g (g^*)^{-1} \circ P_g$, where the coefficients P_g rapidly decay. We should prove that

1. Each function $\text{tr}(g^*)^{-1} \circ P_g \Delta^{-z}$ admits a meromorphic continuation into the entire complex plane with the same discrete set of possible poles.

2. The series formed by these functions locally uniformly converges outside this set, thus giving a meromorphic function as the sum.

To this end, one uses the representation

$$\Delta^{-z} = \frac{1}{2\pi i} \int_{\gamma} \lambda^{-z} (\lambda - \Delta)^{-1} dz = \frac{l!}{(1-z) \cdots (l-z)} \int_{\gamma} \lambda^{l-z} (\lambda - \Delta)^{-1-l} dz,$$

where γ is a contour going along the imaginary axis and bypassing the origin on the right. For sufficiently large l , the operator $(g^*)^{-1} \circ P_g (\lambda - \Delta)^{-1-l}$ is trace class, and the integral converges and defines an analytic function for $\text{Re } z \gg 0$. To get the desired analytic continuation, we note that $(\lambda - \Delta)^{-1-l}$ is a pseudodifferential operator with parameter λ in the sense of [12] and hence has the complete symbol possessing an asymptotic expansion in homogeneous functions of integer orders of homogeneity in (ξ, λ) , where ξ are the covariables. In the trace integral, only an arbitrarily small neighborhood of $\text{fix } g$ can produce terms that are not extendable as entire functions; in these neighborhoods, the stationary phase method reduces the integrals involving the homogeneous components of the symbol over the phase space to asymptotic expansions in (half)-integer powers of λ whose coefficients are integrals, independent of λ , over the cotangent bundles of M_g . The subsequent integration over γ gives functions of z with simple poles at half-integers in some left half-plane. Finally, a careful estimate of the size of neighborhoods and remainders of the stationary phase method shows that under the Diophantine condition the series over g converges as desired. \square

2.2.2 The odd case

Now let X be an odd-dimensional spin manifold with an isometric action of Γ such that all conditions from the preceding subsection are satisfied. Let

$$\sigma \in \text{Mat}(N, C^\infty(X)_\Gamma)$$

be an invertible element. Consider the Toeplitz operator

$$P\sigma P : PL^2(X, S \otimes \mathbb{C}^N) \longrightarrow PL^2(X, S \otimes \mathbb{C}^N),$$

where P is the nonnegative spectral projection of the Dirac operator \mathcal{D}_X .

Theorem 8. *One has*

$$\text{ind } P\sigma P = \sum_g \left[\sum_{X_g} \int_{\dot{X}_g} A(X_g) \text{Pf} \left\{ 2i \sin \left(\frac{\Omega}{4\pi} + \frac{i\Theta}{2} \right) (NX_g) \right\}^{-1} \text{ch } \sigma(g) \right], \quad (12)$$

where $\text{ch } \sigma(g)$ is the coefficient of g^* in the Chern–Simons character of the invertible function σ .

Proof. This formula can be obtained from the corresponding even formula obtained above by the standard trick (multiplication of the manifold by \mathbb{S}^1 ; cf. [13]). \square

2.3 End of proof of the main theorem

Now we can apply the results obtained in the preceding subsection and prove the index formula.

Note that, assuming that M is orientable, the cotangent bundle of M is a spin manifold; for the spin bundle one can take the complexification of the lift of the exterior form bundle from M to T^*M . In this case, the Todd operator is a Dirac operator.

Then the index formula (12) gives

$$\begin{aligned} \text{ind } P\sigma(D)|_{S^*M} P \\ = \sum_g \left[\sum_{(S^*M)_g} \int_{(S^*M)_g} \frac{A((S^*M)_g) \text{ch}_\Gamma \sigma(D)|_{S^*M}(g)}{\text{Pf} \left\{ 2i \sin \left(\frac{\Omega}{4\pi} + \frac{i\Theta}{2} \right) (N(S^*M)_g) \right\}} \right]. \end{aligned} \quad (13)$$

In our case, this gives the desired formula (7). Indeed,

1) The fixed point set $(S^*M)_g$ is the cosphere bundle S^*M_g of the corresponding fixed point set in M .

2) For the A -class of the fixed point set, one has (e.g., see [14])

$$A(S^*M_g) = A(T^*M_g) = A(M_g)^2 = \text{Td}(TM_g \otimes \mathbb{C})$$

(where equality holds not only for characteristic classes but also for the corresponding differential forms).

3) Finally,

$$\text{ch } \lambda_{-1}(NM_g \otimes \mathbb{C})(g) = \text{Pf} \left\{ 2i \sin \left(\frac{\Omega}{4\pi} + \frac{i\Theta}{2} \right) (N(S^*M)_g) \right\}.$$

This can be proved by more or less routine computations, which will be given in the detailed version of the paper.

The proof of the index theorem is complete.

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